CARIBBEAN EXAMINATIONS COUNCIL

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 1 – Paper 02

ALGEBRA, GEOMETRY AND CALCULUS

2 hours 30 minutes

14 MAY 2013 (p.m.)

This examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 6 printed pages.

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

SECTION A (Module 1)

Answer BOTH questions.

1. (a) Let **p** and **q** be two propositions. Construct a truth table for the statements

(i) $p \rightarrow q$ **[1 mark]**

(ii) $\sim (p \wedge q)$. **[2 marks]**

(b) A binary operator $\oplus$ is defined on a set of positive real numbers by

$$y \oplus x = y^2 + x^2 + 2y + x - 5xy.$$

Solve the equation $2 \oplus x = 0$. **[5 marks]**

(c) Use mathematical induction to prove that $5^n + 3$ is divisible by 2 for all values of $n \in \mathbf{N}$. **[8 marks]**

(d) Let $f(x) = x^3 - 9x^2 + px + 16$.

(i) Given that $(x + 1)$ is a factor of $f(x)$, show that $p = 6$. **[2 marks]**

(ii) Factorise $f(x)$ completely. **[4 marks]**

(iii) Hence, or otherwise, solve $f(x) = 0$. **[3 marks]**

Total 25 marks

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2. (a) Let $A = \{x : x \in \mathbf{R}, x \geq 1\}$.

A function $f : A \rightarrow \mathbf{R}$ is defined as $f(x) = x^2 - x$. Show that f is one to one. **[7 marks]**

(b) Let $f(x) = 3x + 2$ and $g(x) = e^{2x}$.

(i) Find

a) $f^{-1}(x)$ and $g^{-1}(x)$ **[4 marks]**

b) $f[g(x)]$ (or $f \circ g(x)$). **[1 mark]**

(ii) Show that $(f \circ g)^{-1}(x) = g^{-1}(x) \circ f^{-1}(x)$. **[5 marks]**

(c) Solve the following:

(i) $3x^2 + 4x + 1 \leq 5$ **[4 marks]**

(ii) $|x + 2| = 3x + 5$ **[4 marks]**

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) (i) Show that $\sin 2\theta = \dfrac{2 \tan \theta}{1 + \tan^2 \theta}$. **[4 marks]**

(ii) Hence, or otherwise, solve $\sin 2\theta - \tan \theta = 0$ for $0 \leq \theta \leq 2\pi$. **[8 marks]**

(b) (i) Express $f(\theta) = 3 \cos \theta - 4 \sin \theta$ in the form $r \cos (\theta + \alpha)$ where

$r > 0$ and $0° \leq \alpha \leq \dfrac{\pi}{2}$. **[4 marks]**

(ii) Hence, find

a) the maximum value of $f(\theta)$ **[2 marks]**

b) the minimum value of $\dfrac{1}{8 + f(\theta)}$. **[2 marks]**

(iii) Given that the sum of the angles A, B and C of a triangle is π radians, show that

a) $\sin A = \sin (B + C)$ **]3 marks]**

b) $\sin A + \sin B + \sin C = \sin (A + B) + \sin (B + C) + \sin (A + C)$. **[2 marks]**

Total 25 marks

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4. (a) A circle C is defined by the equation $x^2 + y^2 - 6x - 4y + 4 = 0$.

(i) Show that the centre and the radius of the circle, C, are (3, 2) and 3, respectively. **[3 marks]**

(ii) a) Find the equation of the normal to the circle C at the point (6, 2). **[3 marks]**

b) Show that the tangent to the circle at the point (6, 2) is parallel to the y-axis. **[3 marks]**

(b) Show that the Cartesian equation of the curve that has the parametric equations

$$x = t^2 + t, \; y = 2t - 4$$

is $4x = y^2 + 10y + 24$. **[4 marks]**

(c) The points A (3, –1, 2), B (1, 2, –4) and C (–1, 1, –2) are three vertices of a parallelogram $ABCD$.

(i) Express the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. **[3 marks]**

(ii) Show that the vector $\mathbf{r} = -16\mathbf{j} - 8\mathbf{k}$ is perpendicular to the plane through A, B and C. **[5 marks]**

(iii) Hence, find the Cartesian equation of the plane through A, B and C. **[4 marks]**

Total 25 marks

SECTION C (Module 3)

Answer BOTH questions.

5. (a) A function $f(x)$ is defined as $f(x) = \begin{cases} x + 2, & x < 2 \\ x^2, & x > 2 \end{cases}$.

(i) Find $\lim_{x \to 2} f(x)$. **[4 marks]**

(ii) Determine whether $f(x)$ is continuous at $x = 2$. **Give a reason for your answer.** **[2 marks]**

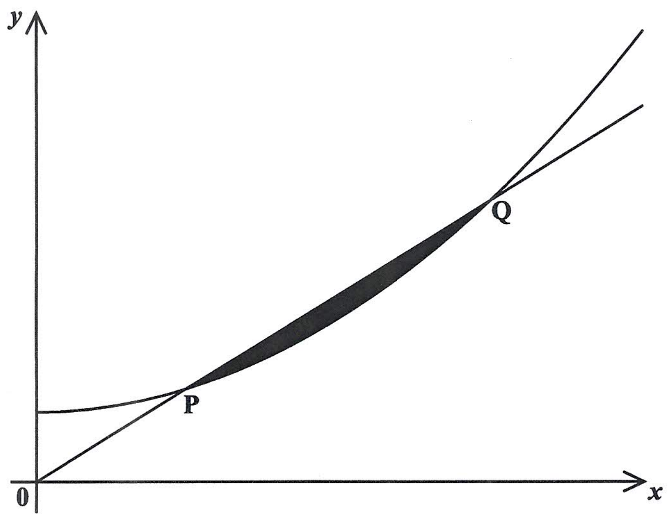
(b) Let $y = \dfrac{x^2 + 2x + 3}{(x^2 + 2)^3}$. Show that $\dfrac{dy}{dx} = \dfrac{-4x^3 - 10x^2 - 14x + 4}{(x^2 + 2)^4}$. **[5 marks]**

(c) The equation of an ellipse is given by

$$x = 1 - 3\cos\theta, \quad y = 2\sin\theta, \quad 0 \le \theta \le 2\pi.$$

Find $\dfrac{dy}{dx}$ in terms of θ. **[5 marks]**

(d) The diagram below (**not drawn to scale**) shows the curve $y = x^2 + 3$ and the line $y = 4x$.



(i) Determine the coordinates of the points **P** and **Q** at which the curve and the line intersect. **[4 marks]**

(ii) Calculate the area of the shaded region. **[5 marks]**

Total 25 marks

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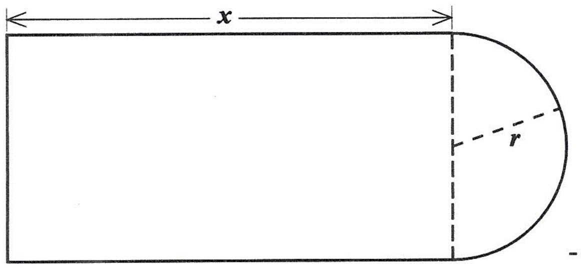
6. (a) (i) By using the substitution $u = 1 - x$, find $\int x(1-x)^2\,dx$. **[5 marks]**

(ii) Given that $f(t) = 2\cos t$, $g(t) = 4\sin 5t + 3\cos t$,

show that $\int [f(t) + g(t)]\,dt = \int f(t)\,dt + \int g(t)\,dt$. **[4 marks]**

(b) A sports association is planning to construct a running track in the shape of a rectangle surmounted by a semicircle, as shown in the diagram below. The letter x represents the length of the rectangular section and r represents the radius of the semicircle.



The perimeter of the track must be 600 metres.

(i) Show that $r = \dfrac{600 - 2x}{2 + \pi}$. **[2 marks]**

(ii) Hence, determine the length, x, that **maximises** the area enclosed by the track. **[6 marks]**

(c) (i) Let $y = -x\sin x - 2\cos x + Ax + B$, where A and B are constants.

Show that $y'' = x\sin x$. **[4 marks]**

(ii) Hence, determine the specific solution of the differential equation

$$y'' = x\sin x,$$

given that when $x = 0$, $y = 1$ and when $x = \pi$, $y = 6$. **[4 marks]**

Total 25 marks

END OF TEST

FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.