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FORM TP 2014240

MAY/JUNE 2014

CARIBBEAN EXAMINATIONS COUNCIL

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 1 – Paper 02

ALGEBRA, GEOMETRY AND CALCULUS

2 hours 30 minutes

13 MAY 2014 (p.m.)

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Answer ALL questions from the THREE sections.
3. Each section consists of TWO questions.
4. Write your solutions, with full working, in the answer booklet provided.
5. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

SECTION A

Module 1

Answer BOTH questions.

1. (a) Let **p**, **q** and **r** be three propositions. Construct a truth table for the statement

$$(\mathbf{p} \rightarrow \mathbf{q}) \wedge (\mathbf{r} \rightarrow \mathbf{q}).$$ **[5 marks]**

(b) A binary operator $\oplus$ is defined on a set of positive real numbers by

$$y \oplus x = y^3 + x^3 + ay^2 + ax^2 - 5y - 5x + 16 \text{ where } a \text{ is a real number.}$$

(i) State, giving a reason for your answer, if $\oplus$ is commutative in **R**. **[3 marks]**

(ii) Given that $f(x) = 2 \oplus x$ and $(x - 1)$ is a factor of $f(x)$,

a) find the value of a **[4 marks]**

b) factorize $f(x)$ completely. **[3 marks]**

(c) Use mathematical induction to prove that

$$1^2 + 3^2 + 5^2 + ... + (2n - 1)^2 = \frac{n}{3}(4n^2 - 1) \text{ for } n \in \mathbf{N}.$$ **[10 marks]**

Total 25 marks

2. (a) The functions f and g are defined as follows:

$$f(x) = 2x^2 + 1$$
$$g(x) = \sqrt{\frac{x-1}{2}} \qquad \text{where } 1 \leq x < \infty,\ x \in \mathbf{R}.$$

(i) Determine, in terms of x,

a) $f^2(x)$ **[3 marks]**

b) $f[g(x)]$. **[3 marks]**

(ii) Hence, or otherwise, state the relationship between f and g. **[1 mark]**

(b) Given that $a^3 + b^3 + 3a^2b = 5ab^2$, show that $3 \log \left(\frac{a+b}{2}\right) = \log a + 2 \log b$. **[5 marks]**

(c) Solve EACH of the following equations:

(i) $e^x + \frac{1}{e^x} - 2 = 0$ **[4 marks]**

(ii) $\log_2 (x + 1) - \log_2 (3x + 1) = 2$ **[4 marks]**

(d) Without the use of a calculator, show that

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{2}-1}{\sqrt{2}+1} + \frac{\sqrt{2}+1}{\sqrt{2}-1} = 10.$$ **[5 marks]**

Total 25 marks

SECTION B

Module 2

Answer BOTH questions.

3. (a) (i) Prove that $\dfrac{\cot y - \cot x}{\cot x + \cot y} = \dfrac{\sin(x-y)}{\sin(x+y)}$. **[4 marks]**

(ii) Hence, or otherwise, find the possible values for y in the trigonometric equation

$$\frac{\cot y - \cot x}{\cot x + \cot y} = 1, \ 0 \le y \le 2\pi,$$

when $\sin x = \dfrac{1}{2}$, $0 \le x \le \dfrac{\pi}{2}$. **[8 marks]**

(b) (i) Express $f(\theta) = 3 \sin 2\theta + 4 \cos 2\theta$ in the form $r \sin(2\theta + \alpha)$ where

$r > 0$ and $0 < \alpha < \dfrac{\pi}{2}$. **[4 marks]**

(ii) Hence, or otherwise, determine

a) the value of θ, between 0 and 2π radians, at which $f(\theta)$ is a **minimum** **[4 marks]**

b) the minimum and maximum values of $\dfrac{1}{7 - f(\theta)}$. **[5 marks]**

Total 25 marks

4. (a) Let L_1 and L_2 be two diameters of a circle C. The equations of L_1 and L_2 are $x - y + 1 = 0$ and $x + y - 5 = 0$, respectively.

(i) Show that the coordinates of the centre of the circle, C, where L_1 and L_2 intersect are (2, 3). **[3 marks]**

(ii) A and B are endpoints of the diameter L_1. Given that the coordinates of A are (1, 2) and that the diameters of a circle **bisect** each other, determine the coordinates of B. **[3 marks]**

(iii) A point, p, moves in the $x - y$ plane such that its distance from C (2, 3) is always $\sqrt{2}$ units. Determine the locus of p. **[3 marks]**

(b) The parametric equations of a curve, S, are given by

$$x = \frac{1}{1+t} \text{ and } y = \frac{t}{1-t^2}.$$

Determine the Cartesian equation of the curve, S. **[6 marks]**

(c) The points P (3, –2, 1), Q (–1, λ, 5) and R (2, 1, –4) are three vertices of a triangle PQR.

(i) Express EACH of the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RP}$ in the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. **[4 marks]**

(ii) Hence, find the value of λ, given that PQR is right-angled with the side PQ as hypotenuse. **[6 marks]**

Total 25 marks

SECTION C

Module 3

Answer BOTH questions.

5. (a) Let f(x) be a function defined as

$$f(x) = \begin{cases} ax + 2, & x < 3 \\ ax^2, & x \geq 3 \end{cases}.$$

(i) Find the value of a if f(x) is continuous at $x = 3$. **[4 marks]**

(ii) Let $g(x) = \dfrac{x^2 + 2}{bx^2 + x + 4}$.

Given that $\lim_{x \to 1} 2g(x) = \lim_{x \to 0} g(x)$, find the value of b. **[5 marks]**

(b) (i) Let $y = \dfrac{1}{\sqrt{x}}$. Using first principles, find $\dfrac{dy}{dx}$. **[8 marks]**

(ii) If $y = \dfrac{x}{\sqrt{1 + x}}$, determine an expression for $\dfrac{dy}{dx}$.

Simplify the answer FULLY. **[4 marks]**

(c) The parametric equations of a curve are given by

$x = \cos\theta,\ y = \sin\theta,\ 0 \leq \theta \leq 2\pi$.

Find $\dfrac{dy}{dx}$ in terms of θ.

Simplify the answer as far as possible. **[4 marks]**

Total 25 marks

6. (a) The gradient of a curve which passes through the point (–1, –4) is given by

$$\frac{dy}{dx} = 3x^2 - 4x + 1.$$

(i) Find

a) the equation of the curve **[4 marks]**

b) the coordinates of the stationary points and determine their nature. **[8 marks]**

(ii) Sketch the curve in (a) (i) a) above, clearly marking ALL stationary points and intercepts. **[4 marks]**

(b) The equation of a curve is given by $f(x) = 2x\sqrt{1 + x^2}$.

(i) Evaluate $\int_0^3 f(x)\, dx$. **[5 marks]**

(ii) Find the volume generated by rotating the area bounded by the curve in (b) (i) above, the x-axis, and the lines $x = 0$ and $x = 2$ about the x-axis. **[4 marks]**

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.