C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 1 – Paper 02

ALGEBRA, GEOMETRY AND CALCULUS

2 hours 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of TWO questions.
3. Answer ALL questions from the THREE sections.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra lined page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

SECTION A

Module 1

Answer BOTH questions.

1. (a) Let $f(x) = 2x^3 - x^2 + px + q$.

(i) Given that $x + 3$ is a factor of $f(x)$ and that there is a remainder of 10, when $f(x)$ is divided by $x + 1$ show that $p = -25$ and $q = -12$.

[7 marks]

(ii) Hence, solve the equation $f(x) = 0$.

[6 marks]

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(b) Use mathematical induction to prove that $6^n - 1$ is divisible by 5 for all natural numbers n.

[6 marks]

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(c) (i) Given that **p** and **q** are two propositions, complete the truth table below:

p	q	$p \rightarrow q$	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow (p \wedge q)$
T	T				
T	F				
F	T				
F	F				

[4 marks]

(ii) State, giving a reason for your response, whether the following statements are logically equivalent:

- $p \rightarrow q$
- $(p \vee q) \rightarrow (p \wedge q)$

....

....

[2 marks]

Total 25 marks

2. (a) Solve the following equation for x:

$$\log_2 (10 - x) + \log_2 x = 4$$

[6 marks]

(b) A function f is defined by $f(x) = \dfrac{x+3}{x-1}$, $x \neq 1$.

Determine whether f is bijective, that is, both one-to-one and onto.

(c) Let the roots of the equation $2x^3 - 5x^2 + 4x + 6 = 0$ be α, β and γ.

(i) State the values of $\alpha + \beta + \gamma$, $\alpha\beta + \alpha\gamma + \beta\gamma$ and $\alpha\beta\gamma$.

[3 marks]

(ii) Hence, or otherwise, determine an equation with integer coefficients which has roots $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$.

Note: $(\alpha\beta)^2 + (\alpha\gamma)^2 + (\beta\gamma)^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma\,(\alpha + \beta + \gamma)$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\,(\alpha\beta + \alpha\gamma + \beta\gamma)$

SECTION B

Module 2

Answer BOTH questions.

3. (a) (i) Show that $\sec^2\theta = \dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta}$.

[4 marks]

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(ii) Hence, or otherwise, solve the equation $\dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta} = \dfrac{4}{3}$ for $0 \le \theta \le 2\pi$.

[5 marks]

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(b) (i) Express the function $f(\theta) = \sin \theta + \cos \theta$ in the form $r \sin (\theta + \alpha)$, where $r > 0$ and $0 \leq \theta \leq \frac{\pi}{2}$.

[5 marks]

(ii) Hence, find the maximum value of f and the **smallest** non-negative value of θ at which it occurs.

[5 marks]

(c) Prove that

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}.$$

4. (a) (i) Given that $\sin \theta = x$, show that $\tan \theta = \dfrac{x}{\sqrt{1-x^2}}$, where $0 < \theta < \dfrac{\pi}{2}$.

[3 mark

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(ii) Hence, or otherwise, determine the Cartesian equation of the curve defined parametrically by $y = \tan 2t$ and $x = \sin t$ for $0 < t < \frac{\pi}{2}$.

[5 marks]

(b) Let $\mathbf{u} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ be two position vectors in $\mathbf{R}^3$.

(i) Calculate the lengths of **u** and **v** respectively.

[3 marks]

(ii) Find $\cos\theta$ where θ is the angle between **u** and **v** in $\mathbf{R}^3$.

[4 marks]

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(c) A point, $P(x, y)$, moves such that its distance from the x-axis is half its distance from the origin.

Determine the Cartesian equation of the locus of P.

[5 marks]

(d) The line L has the equation $2x + y + 3 = 0$ and the circle C has the equation $x^2 + y^2 = 9$. Determine the points of intersection of the circle C and the line L.

[5 marks]

Total 25 marks

SECTION C

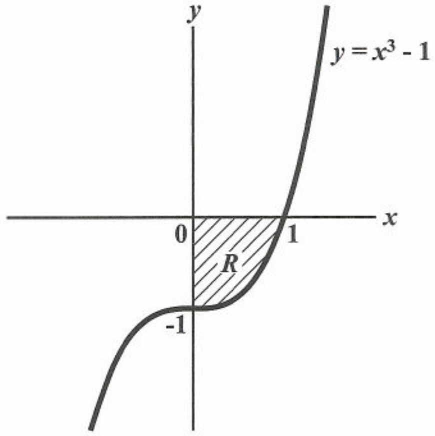
Module 3

Answer BOTH questions.

5. (a) Use an appropriate substitution to find $\int (x+1)^{\frac{1}{3}}\ dx$.

[4 marks]

(b) The diagram below represents the finite region R which is enclosed by the curve $y = x^3 - 1$ and the lines $x = 0$ and $y = 0$.



Calculate the volume of the solid that results from rotating R about the y-axis.

[5 marks]

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(c) Given that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \quad a > 0$, show that

$$\int_0^1 \frac{e^x}{e^x + e^{1-x}}\,dx = \frac{1}{2}.$$

[6 marks]

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(d) An initial population of 10 000 bacteria grow exponentially at a rate of 2% per hour, where $y = f(t)$ is the number of bacteria present t hours later.

(i) Solve an appropriate differential equation to show that the number of bacteria present at any time can be modelled by the equation $y = 10\ 000\ e^{0.02t}$.

[7 marks]

(ii) Determine the time required for the bacteria population to double in size.

[3 marks]

Total 25 marks

6. (a) Find the equation of the tangent to the curve $f(x) = 2x^3 + 5x^2 - x + 12$ at the point where $x = 3$.

[4 marks]

(b) A function f is defined on **R** as

$$f(x) = \begin{cases} x^2 + 2x + 3 & x \leq 0 \\ ax + b & x > 0 \end{cases}$$

(i) Calculate the $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$.

[4 marks]

(ii) Hence, determine the values of a and b such that $f(x)$ is continuous at $x = 0$.

[5 marks]

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(iii) If the value of $b = 3$, determine a such that $f'(0) = \lim_{t \to 0} \dfrac{f(0 + t) - f(0)}{t}$

[6 marks]

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(c) Use first principles to differentiate $f(x) = \sqrt{x}$ with respect to x.

[6 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.