CARIBBEAN EXAMINATIONS COUNCIL

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 2 – Paper 02

ANALYSIS, MATRICES AND COMPLEX NUMBERS

2 hours 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of TWO questions.
3. Answer ALL questions from the THREE sections.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra lined page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

SECTION A

Module 1

Answer BOTH questions.

1. (a) Find the first derivative of the function $f(x) = \cos^{-1}(\sin^{-1} x)$.

[3 marks]

(b) A function, w, is defined as $w(x, y) = \ln \left| \frac{2x + y}{x - 1} \right|$.

(i) Given that $\frac{\partial w}{\partial x} = -\frac{1}{9}$ at the point $(4, y_0)$, calculate the value of y_0.

[5 marks]

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(ii) Show that $\dfrac{\partial^2 w}{\partial y \partial x} - 2\dfrac{\partial^2 w}{\partial y^2} = 0$.

[5 marks]

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(c) (i) Find the complex numbers $u = x + iy$ such that x and y are real and $u^2 = -15 + 8i$.

[7 marks]

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(ii) Hence, or otherwise, solve the equation $z^2 - (3 + 2i)z + (5 + i) = 0$, for z.

[5 marks]

Total 25 marks

2. (a) (i) Use integration by parts to derive the reduction formula

$$aI_n = x^n e^{ax} - nI_{n-1} \text{ where } I_n = \int x^n e^{ax}\, dx .$$

[4 marks]

(ii) Hence, or otherwise, determine $\int x^3 e^{3x}\, dx$.

[6 marks]

(b) Calculate $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$.

[5 marks]

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(c) (i) Use partial fractions to show that

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4}.$$

[5 marks]

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(ii) Hence, or otherwise, determine $\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$.

[5 marks]

Total 25 marks

SECTION B

Module 2

Answer BOTH questions.

3. (a) (i) Determine the Taylor series expansion about $x = 2$ of the function $f(x) = \ln(5 + x)$ up to and including the term in x^3.

[6 marks]

(ii) Hence, obtain an approximation for $f(7) - \ln(7)$.

[2 marks]

(b) (i) Use mathematical induction to prove that

$$1^3 + 2^3 + \ldots + n^3 = \frac{1}{4} n^2 (n + 1)^2, \text{ for } n \in \mathbf{N}.$$

[9 marks]

(ii) Hence, or otherwise, show that $\sum_{i=1}^{2n+1} i^3 = (2n+1)^2 (n+1)^2$.

[3 marks]

(iii) Use the results of Parts (b) (i) and (ii) to show that

$$\sum_{i=1}^{n+1} (2i-1)^3 = (n+1)^2 (2n^2 + 4n + 1) .$$

[5 marks]
Total 25 marks

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4. (a) Eight boys and two girls are to be seated on a bench. How many seating arrangements are possible if the girls can neither sit together nor sit at the ends?

[5 marks]

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(b) (i) Show that the binomial expansion of $(1 + \frac{1}{8}x)^8$ up to and including the term in x^4 is

$$1 + x + \frac{7}{16}x^2 + \frac{7}{64}x^3 + \frac{35}{2048}x^4 .$$

[4 marks]

(ii) Use the expansion to approximate the value of $(1.0125)^8$.

[4 marks]

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(c) (i) Use the intermediate value theorem to show that $f(x) = \sqrt{x} - \cos x$ has a root in the interval [0, 1].

[3 marks]

(ii) Use two iterations of the interval bisection method to approximate the root of f in the interval [0, 1].

[4 marks]

(d) (i) Show that $x_{n+1} = \sqrt[3]{\frac{9-3x_n}{2}}$ is an appropriate iterative formula for finding the root of $f(x) = -2x^3 - 3x + 9$.

[2 marks]

(ii) Apply the iterative formula with initial approximation $x_1 = 1$, to obtain a third approximation, x_3, of the root of the equation.

[3 marks]

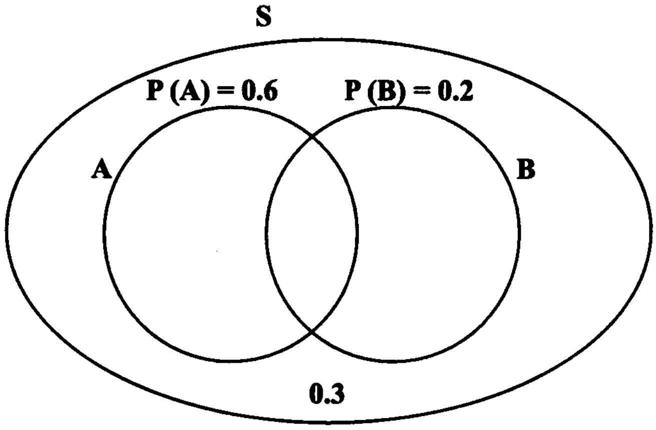
Total 25 marks

SECTION C

Module 3

Answer BOTH questions.

5. (a) The following Venn diagram shows a sample space, S, and the probabilities of two events, A and B, within the sample space S.



(i) Given that $P(A \cup B) = 0.7$, calculate P (A only).

[3 marks]

(ii) Hence, determine whether events A and B are independent. **Justify your answer**.

[2 marks]

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(b) Two balls are to be drawn at random without replacement from a bag containing 3 red balls, 2 blue balls and 1 white ball.

(i) Represent the outcomes of the draws and their corresponding probabilities on a tree diagram.

[4 marks]

(ii) Determine the probability that the second ball drawn is white.

[3 marks]

(c) Given the matrices $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 30 & -12 & 2 \\ 5 & -8 & 3 \\ -5 & 4 & 1 \end{pmatrix}$,

(i) show that $AB = 20I$.

[5 marks]

(ii) Hence, deduce the inverse, A^{-1}, of the matrix A.

[2 marks]

(iii) Hence, or otherwise, solve the system of linear equations given by

$$\begin{aligned} x - y + z &= 1 \\ x - 2y + 4z &= 5 \\ x + 3y + 9z &= 25 \end{aligned}$$

[6 marks]

Total 25 marks

6. (a) (i) Find the general solution of the differential equation $(1 + x^2)\dfrac{dy}{dx} + 2xy = \sqrt[3]{x}$.

[7 marks]

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(ii) Hence, given that $y = 2$ when $x = 0$, calculate $y(1)$.

[3 marks]

(b) (i) Use the substitution $u = y'$ to show that the differential equation $y'' + 4y' = 2\cos 3x - 4\sin 3x$ can be reduced to $u' + 4u = 2\cos 3x - 4\sin 3x$.

[2 marks]

(ii) Hence, or otherwise, find the general solution of the differential equation.

[13 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.