CARIBBEAN EXAMINATIONS COUNCIL

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 2 – Paper 02

ANALYSIS, MATRICES AND COMPLEX NUMBERS

2 hours 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of TWO questions.
3. Answer ALL questions from the THREE sections.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

SECTION A

Module 1

Answer BOTH questions.

1. (a) (i) A curve P is defined parametrically as $x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$.

Determine the gradient of the curve at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

[5 marks]

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(ii) Hence, or otherwise, determine the x and y intercepts of the tangent that touches the curve at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

[4 marks]

(b) Let the function $f(x, y) = \sin(kx)\sin(aky)$. Determine $\dfrac{\partial^2 f(x, y)}{\partial x\, \partial y}$.

[3 marks]

(c) Use DeMoivre's theorem to show that $\sin 5\theta = \sin^5 \theta - 10 \sin^3 \theta \cos^2 \theta + 5 \cos^4 \theta \sin \theta$.

[5 marks]

(d) (i) Write the complex number $z = (1 - i)$ in the form $re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$.

[3 marks]

(ii) Hence, show that $(1 - i)^9 = 16(1 - i)$.

[5 marks]

Total 25 marks

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2. (a) Determine

(i) $\int x^5 \cos(x^3)\, dx$

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[8 marks]

(ii) $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\, dx.$

[5 marks]

(b) (i) Use partial fractions to show that $\frac{x^4 + 1}{x(x^2 + 1)^2} = \frac{1}{x} - \frac{2x}{(x^2 + 1)^2}$.

[8 marks]

(ii) Hence, determine $\int \frac{x^4 + 1}{x\,(x^2 + 1)^2}\,dx$.

[4 marks]

Total 25 marks

SECTION B

Module 2

Answer BOTH questions.

3. A sequence $\{a_n\}$ is such that $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$.

(a) (i) State the third term, a_3, of the sequence.

[2 marks]

(ii) Use mathematical induction to prove that x_n is increasing and that it is bounded above by 3, that is, $a_n < a_{n+1}$ and $a_n \leq 3$ for all $n \in \mathbf{N}$.

[8 marks]

(b) (i) Let $f(x) = e^{-x^2}$. By calculating the first three non-zero terms and assuming that the pattern continues, show that the Maclaurin series expansion of $f(x)$ may be expressed as $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}$.

[8 marks]

(ii) Hence, or otherwise, determine the values of x for which the expansion is valid.

[3 marks]

(c) Determine the sum of the series $\sum_{n=1}^{\infty}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right)$.

[4 marks]

Total 25 marks

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4. (a) Determine the coefficient of the term in x^7 in the expression $\left(x^2 - \frac{3}{x}\right)^8$.

[4 marks]

(b) By expressing ${}^{n}C_{r}$ and ${}^{n}C_{r-1}$ in terms of factorials, show that ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$.

[6 marks]

(c) (i) Use the intermediate value theorem to show that the equation $4 \cos x - x^3 + 2 = 0$ has a root in the interval (1, 1.5).

[3 marks]

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(ii) Use linear interpolation to approximate the value of the root of the equation $4\cos x - x^3 + 2 = 0$ in the interval (1, 1.5), correct to two decimal places.

[8 marks]

(d) The equation $3e^x = 1 - 2 \ln x$ is known to have a root in the interval (0, 1).

Taking $x_1 = 0.2$ as the first approximation of the root, use the Newton–Raphson method to find a second approximation, x_2, of the root in the interval (0, 1).

[4 marks]

Total 25 marks

SECTION C

Module 3

Answer BOTH questions.

5. (a) Events A and B are such that $P(A) = 0.4$, $P(B) = 0.45$ and $P(A \cup B) = 0.68$.

(i) Calculate $P(A \cap B)$.

[3 marks]

(ii) Determine whether the events A and B are independent. **Justify your response.**

[3 marks]

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(b) A committee of 4 persons is to be chosen from 8 persons, including Mr Smith and his wife. Mr Smith will not join the committee without his wife, but his wife will join the committee without him.

Calculate the number of possible ways the committee of 4 persons can be formed.

[5 marks]

(c) How many odd numbers greater than 500 000 can be made from the digits 2, 3, 4, 5, 6, 7, **without repetitions**?

[5 marks]

(d) Two matrices, are given as

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 3 \\ 0 & 3 & -4 \\ 2 & 0 & 6 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 18 & 12 & -1 \\ -8 & 24 & 20 \\ -6 & -4 & 15 \end{pmatrix}.$$

(i) By finding **AB**, deduce that $\mathbf{A}^{-1} = \frac{1}{88}\,\mathbf{B}$.

[5 marks]

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(ii) Hence, or otherwise, solve the system of equations given as

$$\begin{pmatrix} 5 & -2 & 3 \\ 0 & 3 & -4 \\ 2 & 0 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \\ -6 \end{pmatrix}.$$

[4 marks]

Total 25 marks

6. (a) A differential equation is given as $y' \cos x = y \sin x + \sin 2x$.

(i) Show that the general solution of the differential equation is

$\frac{1}{2} \sec x - \cos x + C \sec x$.

[9 marks]

(ii) Hence, or otherwise, solve the initial value problem
$y' \cos x = y \sin x + \sin 2x, \quad y(0) = 0.$

[2 marks]

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(b) A differential equation is given as $y'' + 2y' + y = xe^{-x}$.

(i) Determine the solution of the complementary equation $y'' + 2y' + y = 0$.

[4 marks]

(ii) Given that the particular solution of a differential equation has the form $y_p = (Ax^3 + Bx^2)e^{-x}$, or otherwise, determine the general solution of the differential equation.

[10 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.